Using analogy in generalization and conceptual learning in computer assisted learning in geometry

Jiří Vaníček
e-mail: vanicek@pf.jcu.cz
University of South Bohemia
České Budějovice, Czech Republic

Abstract

Analogy is one of the methods that can be, with computer support, used for development of the ability to generalize. The paper informs of a project carried out within the frame of pre-service teacher training at University of South Bohemia in České Budějovice. Within this project, technologies were used for visualization of geometrical objects, their mutual relations and dependencies, and that is both between objects in different dimensions and between geometric objects and their algebraic representations.

Using dynamic geometry software Geogebra and Cabri 3D, pre-service teachers of mathematics looked for analogical assignments and construction procedures between 2D and 3D and later also between 3D and 4D. Reactions of pre-service teachers and their ability to look for, pose and solve problems that train analogy was studied.

1. Analogy as a type of generalization

A number of mathematical disciplines, especially algebra and arithmetic, offer many opportunities for generalization of computational algorithms directed from specific numbers to expressions or formulae. In the process of gradual generalization, pupils may proceed from single objects to whole classes, from common properties of different objects to analysis of properties in general. They may study the invariants when the input parameters are changed and thus precise their ideas of the concept under construction.

Development of the ability to generalize forms an important part of an individual's mental development. Sayer stresses that generalization is "an approximate quantitative measure of the numbers of objects belonging to some class or a statement about certain common properties of objects" [9, p. 100]. As far as quantitative analysis and generalization are concerned, Sayer claims that one must ask what properties the objects have in common, in which properties they differ and how many objects have these properties [9].

An under-developed ability to generalize correctly significantly influences an individual's ability to make good decisions in everyday life. There are several methods of improvement of learning of generalization. One of the methods of generalization which may be used by a teacher of mathematics in his/her lessons discussed in this article is analogy. Analogy is a cognitive process of transfer of a parameter of one particular object onto another and verbal expression of this process. In a narrower sense, analogy looks for and describes corresponding relations between two pairs of objects.

Analogy plays a significant role in problem solving, in decision making, perception, memories, creative work, explaining and in communication. It is in the background of problems on recognition of places, objects, people, faces. Some academics argue that analogy is the basis of cognition [4].

An individual's competence to use analogy goes beyond mathematics. Analogy is used in physics, logic, biology, linguistics, in law, legislation etc. The issue of learning mathematics with the help of analogy was explored by e.g. [7, 8]. Mathematics provides a whole range of teaching environments in which the ability to generalize may be developed. This article focuses on the potential of dynamic geometry environment to train generalization by the method of analogy.

Our perspective is anchored in constructivist approach, in which learners take active part in construction of ideas, concepts and relations between them. It seems that one of the mental activities based on this approach is use of analogy as a method in which we study two objects on the basis of the properties they share.

In comparison to other activities in which pupils generalize, analogy may be grasped at a relatively early age. Where analogy is used, it is very often possible to come to an agreement on a specific analogical relation of two pairs of objects. However, to express the principle of this particular analogy verbally is difficult. Teaching analogy leads to training of the ability to express oneself and develops logic. From the point of view of didactics of mathematics, this training may be perceived as introduction to the concept of isomorphism. The oddity of analogy is that it can never be claimed that its use is correct. Thus one must always ask whether the analogy is in place, whether it does not lead to distortion (see Figure 1).

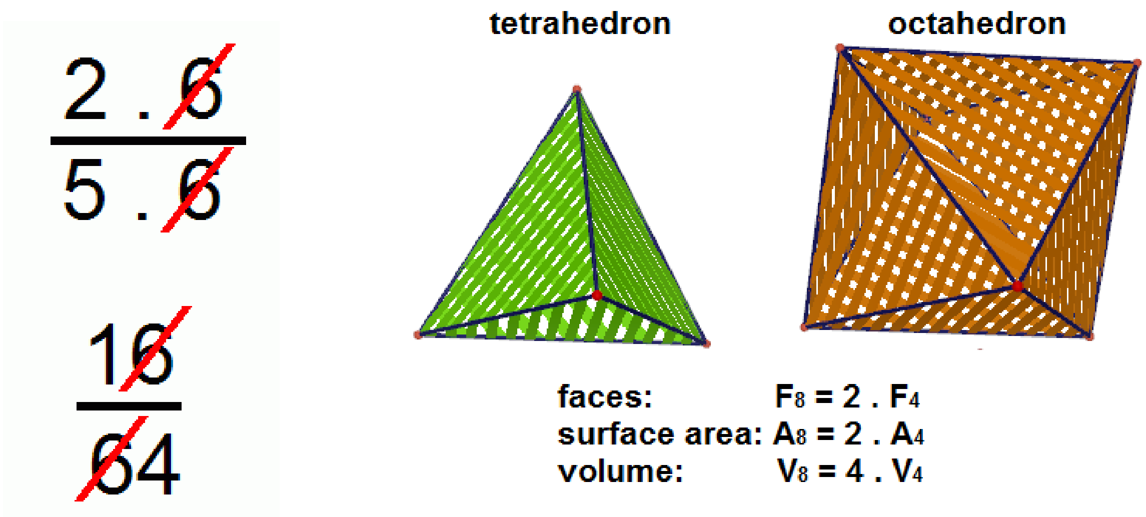

Figure 1: *Examples of inappropriately chosen analogies. On the left reduction of fractions, on the right search for analogies in formulae for volume and surface area in relationship to the number of faces in a tetrahedron and octahedron. In both cases the result is correct but the analogy used is wrong.*

Analogies discussed in this article may not be regarded as basic or simple. Diana Dell lists the most common types of analogies, e.g. synonyms, part and the whole, worker and instrument, masculine and feminine equivalents, measuring and distance, arithmetic relations etc. Analogy between dimensions of space is not on this list [3].

2. Potential of dynamic geometry software in development of analogical thinking

Interactive geometry software Geogebra enables simulation of situations suitable for search for analogies between algebraic expression of a numerical structure and the corresponding geometric representation of such an object. Manipulation facilitated by this software makes it possible to modify the position of point on the screen or to change coordinates by a change of parameters of the relevant algebraic expression. Thus both algebraic way and geometrical manipulation make it possible to change position and shape of objects and make conjectures about invariants, or observe and test behavior of the corresponding geometrical or algebraic object.

Cabri 3D software enables modeling of geometrical figures in space with the help of "spatial" constructive tools. These enable such constructional steps that are, in traditional 2D environment, possible only with special constructional techniques such as tilting, projection onto plane of projection etc. In Cabri 3D it is possible to construct spatial solids with the same procedures and as intuitively as to construct planar objects in 2D software (Cabri II Plus). Combination of such two tools makes it possible to look for analogical construction procedures for construction of analogical figures in 2D and 3D, the use of analogy in dimensional step between planar and spatial geometry for construction of basic notions of 4D objects etc.

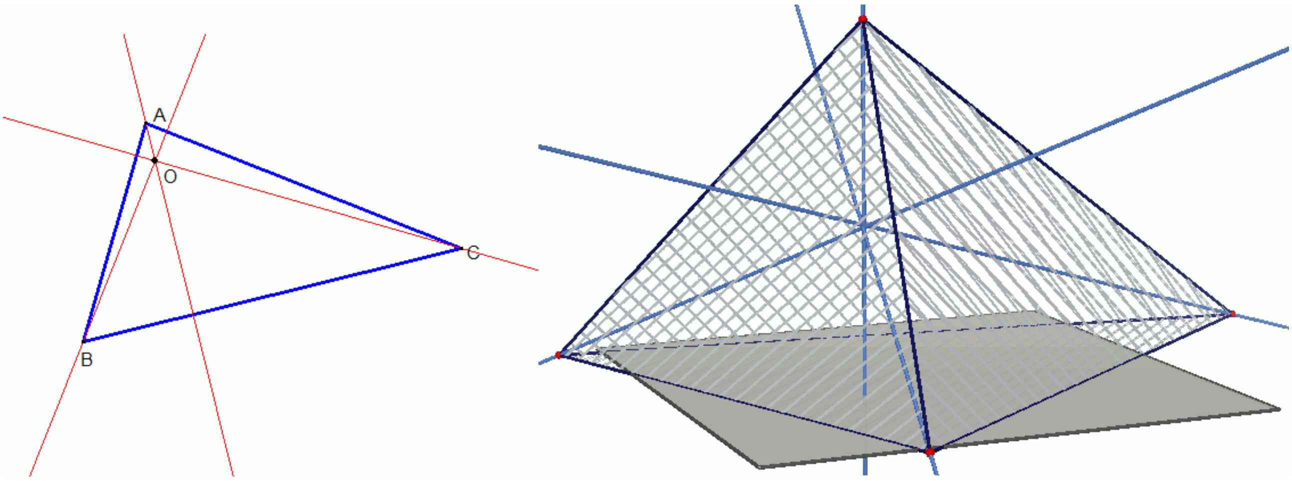

Figure 2: *Orthocenters. If there is an analogy between planar and spatial construction procedure of the orthocenter of a triangle and a tetrahedron, it does not necessarily imply that there is any analogy between the existences of the resulting orthocenters.*

3. Analogy algebra – geometry

The dynamic geometry environment Geogebra uses a very interesting method of construction of relations between objects based on quantitative indices. Unlike other applications, e.g. Cabri, Geogebra makes it possible to transfer measures by direct use of variables in the definition of geometrical objects. While for measurement transfer in Cabri it is necessary to select a particular measured or recorded value, in Geogebra it is sufficient merely to give the name of the object as a parameter of an algebraic expression. In case of a line segment, its length is perceived as its parameter, in case of a point, it is its coordinates. The notation $B=A+a$ may be then understood as "point B whose distance from point A equals to the length of the line segment *a* (for each of the coordinates of this point)".

This interesting feature opens opportunities to use analogy between algebraic expressions and their geometrical representations. E.g. the expression $S_4 = (A+B+C+D)/4$ stands for algebraic mean but on the screen in Geogebra this mean is shown as point *S* lying in the centroid of rectangle *ABCD*.

In the project a cascade of problems offering analogy between geometric and algebraic representation of a relation was used.

1. finding geometric representation of the relation $S_3=(A+B+C)/3$
2. finding parallel analogy for the line segment $S_2=(A+B)/2$
3. finding geometrical representation of the above given relation for S_4 (Figure 3)
4. geometric construction of a centroid of a rectangle to verify the hypothesis from point 3

The described problems were tested on a group of 15 3rd year pre-service teachers of mathematics for primary and lower secondary school level. The students recorded the expression and observed its geometric representation in Geogebra. Having solved the problems with the expression S_2 and S_3 their task was to find the geometric meaning of the expression S_4. Students presumed it would be the centroid. However, very few were able to construct it. The analogy to the expression S_4 failed to help.

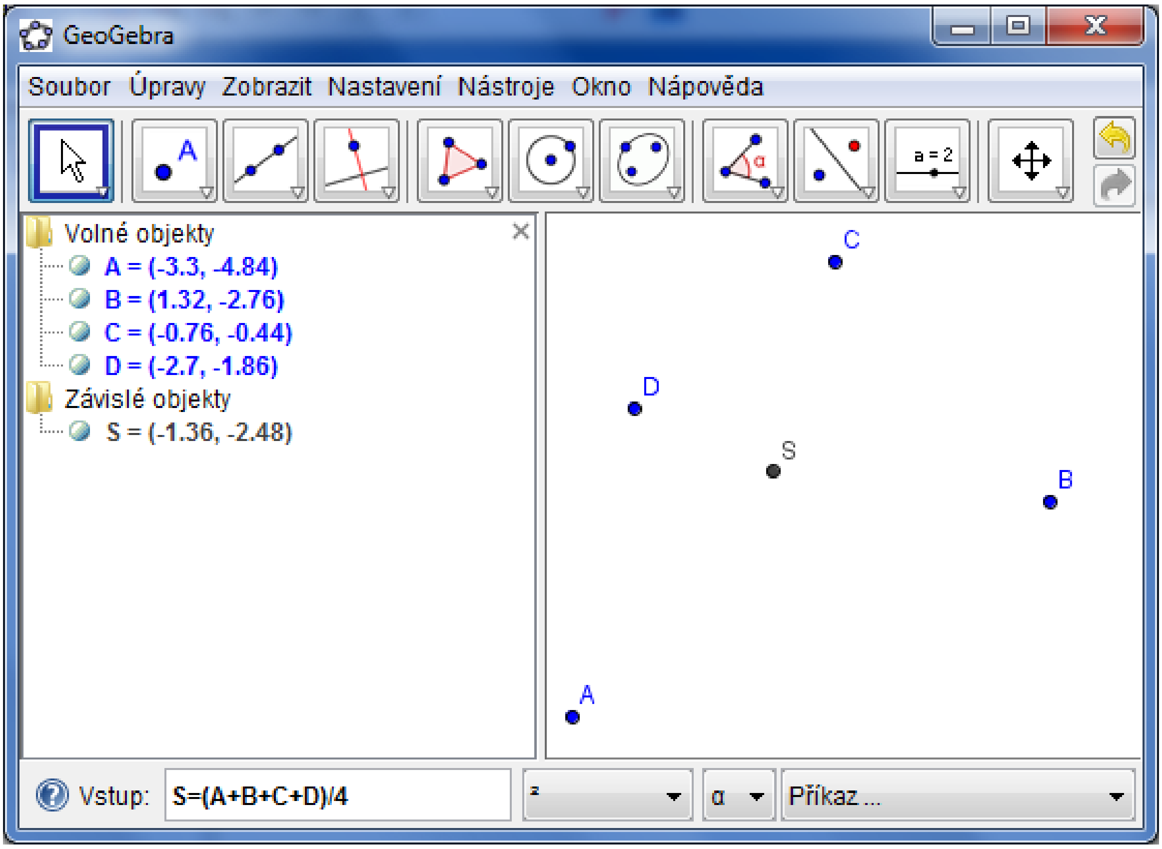

Figure 3: *analogy between algebraic expression and geometric construction on the example of the mean of four numbers and the centroid of a rectangle*

Two students constructed the centroid of a rectangle as the center of the line segment connecting the centers of the opposite sides of rectangle *ABCD*, i.e. join of $(A+B)/2$ and $(C+D)/2$. However, they failed to prove that the thus constructed centroid is point S_4, i.e. geometric representation of a mean (Figure 4). Even when prompted by the lecturer they failed to discover that the formula $((A+B)/2 + (C+D)/2)/2$ after adjustments gives S_4 and that this can serve as a proof. It is highly probable that the students were not familiar with this way of reasoning. It seems they did not master functional connections between forms of geometric and algebraic representations of the same phenomenon and thus did not take in account the possibility to use algebraic representation for the proof of their construction.

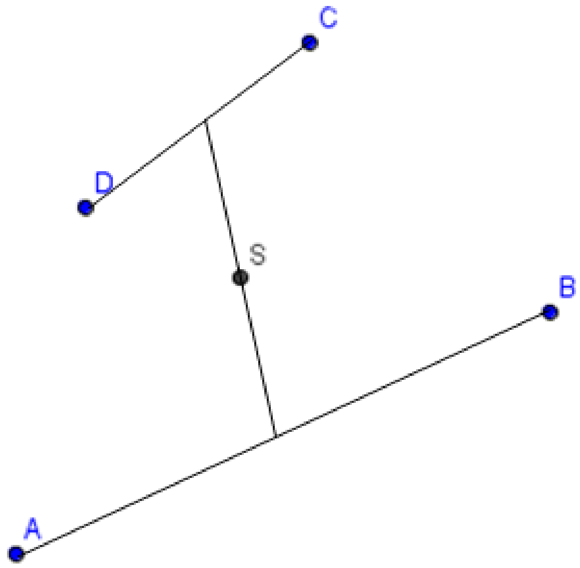

Figure 4: *Construction procedure of looking for the centroid of a rectangle used by the students. Geogebra shows the coalescence of the centroid with geometric representation of a mean.*

4. Analogical problems 2D – 3D

When teaching mathematics at a lower level, analogy is used for comparison of figures and solids, e.g. square and cube, circle and sphere, lines and planes. In our project we focused on analogies between construction procedures in different dimensions.

The following table presents examples of elementary 2D construction problems that are included in curricula for primary and lower secondary Czech schools and their equivalents in 3D. Selected were such problems that not only have analogous assignment but also analogous solution, construction procedure. Assignment of some of these problems may be found in [6].

In Table 1, the assignment and especially the wording are optimized in order to minimalize the differences in descriptions of the corresponding problems. The students were meant to arrive at the stage of this formulation themselves.

Pre-service teachers were asked to look for analogical assignments of problems in 3D analogical to 2D problems. They succeeded in this task. However, they were much less successful when constructing and describing the construction procedure.

E^2	E^3
ASSIGNMENT: Given is a triangle. Construct the circum-circle to this triangle. **SOLUTION:** The center of the circumscribed **circle** lies on the intersection point of **bisectors** of the triangle.	**ASSIGNMENT:** Given is a tetrahedron. Construct the sphere circumscribed to the tetrahedron. **SOLUTION:** The center of the circumscribed **sphere** lies on the intersection point of **planes od symmetry** of the tetrahedron edges.
ASSIGNMENT: Given is a triangle. Find its centroid. **SOLUTION:** The centroid is the intersection point of **medians**. A median is the join of vertex with the midpoint of the opposite **side**.	**ASSIGNMENT:** Given is a tetrahedron. Find its centroid. **SOLUTION:** The centroid is the intersection point of **space medians**. A space median is the join of a vertex with the “midpoint” (i.e. centroid) of the opposite **face**.
ASSIGNMENT: Given are three line segments. Construct a triangle with sides of the given length a, b, c. **SOLUTION:** Construct one line segment. The other vertex is the intersection point of **circles** whose centers are the already constructed vertices and radii the remaining given lengths.	**ASSIGNMENT:** Given are 6 line segments. Construct a tetrahedron with edges of the given length a, b, c, d, e, f. **SOLUTION:** Construct one line segment. The other vertices are the intersection points of **spheres** whose centers are the already constructed vertices and radii the remaining given lengths.
ASSIGNMENT: Given is a circle and point A outside of it. Construct a tangent from point A to the circle. **SOLUTION:** Construct a Thales’ **circle** with the center in the midpoint of the line segment connecting point A and the center of the circle. The tangent point is the intersection point of both **circles**, the tangent is given by point A and the tangent point.	**ASSIGNMENT:** Given is a sphere and point A outside of it. Construct a tangent from point A to the sphere. **SOLUTION:** Construct a Thales’ **sphere** with the center in the midpoint of the line segment connecting point A and the center of the **sphere**. The tangent point is the intersection point of both **spheres**, the tangent is given by point A and the tangent point.
ASSIGNMENT: Given is line segment AB. Construct a square with side of the given line segment. **SOLUTION:** We find another vertex in the intersection point of a suitably selected perpendicular **line** and **circle** with center A and radius AB. The remaining vertex is constructed as an image of the chosen vertex in translation by a vector given by two adjacent vertices.	**ASSIGNMENT:** Given is line segment AB. Construct a cube with edge of the given line segment. **SOLUTION:** We find another vertex in the intersection point of suitably selected perpendicular **plane** and a **sphere** with center A and radius AB. The remaining vertices are constructed as images of chosen vertices in translations by vectors given by two adjacent vertices. **ALTERNATIVE ASSIGNMENT:** *Given is square ABCD. Construct a cube whose face is the given square.*

Table 1: *construction problems in 2D and 3D with analogical assignment and construction procedure*

The assignment of the following problem contained analogical algorithms for constructions in 2D and 3D. The students' task was to examine what the solution of this modified problem would be and whether there would be any analogy in the resulting object. For example in Table 2, the analogical object to the tangent of the circle is the tangent conical surface of the sphere.

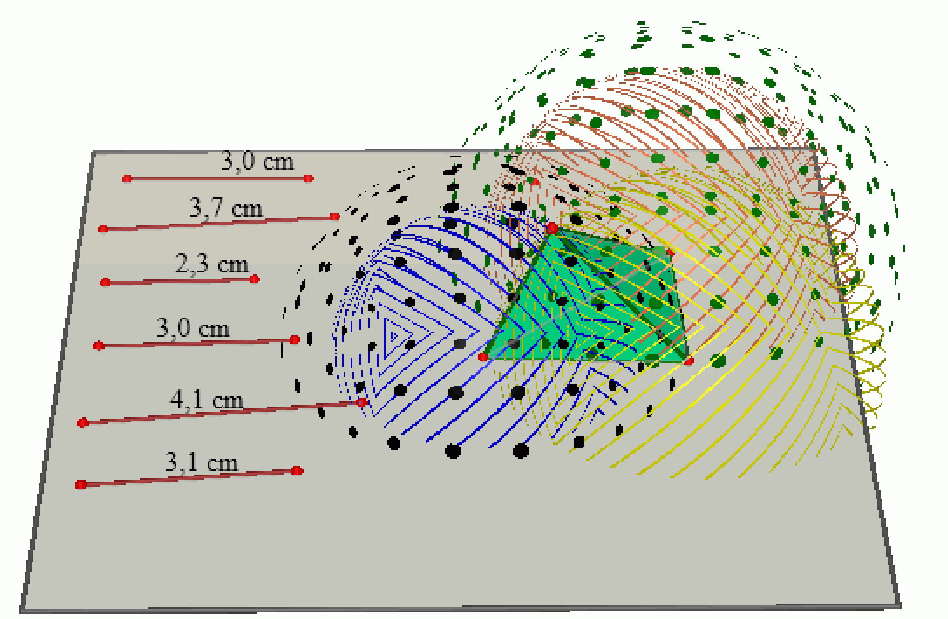

Figure 5: *Analogical construction problem in 3D for triangle of 3 given sides, a student's work*

Analogy may be used not only to study objects and constructions in different dimensions, but also relations between them. An example of this may be an analogy to the condition for existence of a triangle with given 3 sides. Students may examine whether the same condition holds in 3D for tetrahedron and its six faces.

CONSTRUCTION IN 2D: Given is circle k and point A outside it. Construct (Thales') circle with the center in midpoint of the line segment connecting point A and the center of the circle k. Construct the intersection point T of both circles. Construct the join of point A, T.	**CONSTRUCTION IN 3D:** Given is sphere S and point A outside it. Construct (Thales') sphere with the center in midpoint of the line segment connecting point A and the center of the sphere S. Construct the circle of intersection t of both spheres. Construct the surface connecting point A with the circle t.

Table 2: *analogical construction procedure in 2D and 3D resulting in analogical objects*

5. Experiment with pre-service teachers

This year a teaching experiment focusing on looking for analogies between 2D and 3D construction problems was launched within the frame of the course Computer supported mathematics teaching. Students were given several construction problems in plane and were asked to find corresponding solution (in some cases also analogical assignment of the problem) in space. Our experience shows that in some types of problems students find analogical construction process quite easily – especially in case of problems whose single construction steps are familiar to the students from geometry in plane (the midpoint of a line segment, join of two points).

What they found more difficult were problems which needed a construction step for their solution that does not exist in 2D (e.g. plane perpendicular to a line, plane of symmetry of a line segment). The students preferred construction procedure based on familiar steps to the use of analogical step of 3D construction that would have been simpler (Figure 6). E.g. when constructing the vertex of a cube, the students did not use the simpler procedure with looking for intersection of a plane and a sphere. They preferred construction of circles in suitably chosen plane and looked for intersection points of these circles. Their 3D construction procedures then failed to be analogical to 2D procedures and it was difficult to look for analogical procedures.

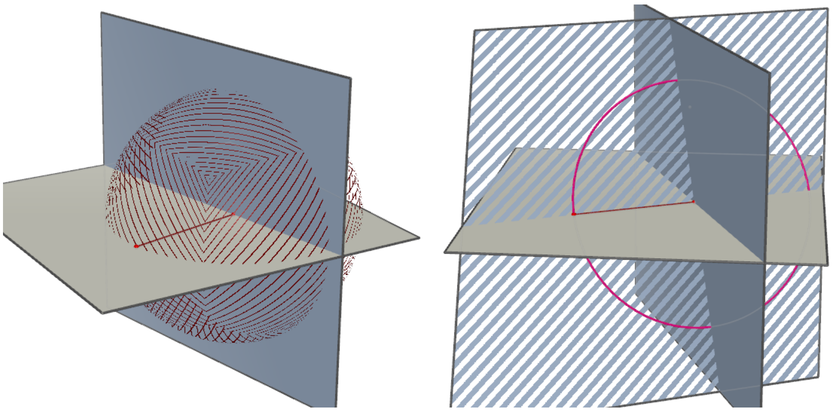

Figure 6: *Transfer of a line segment of a given length to perpendicular plane, construction step carried out in Cabri 3D. Students preferred more complicated procedure using circles and auxiliary plane (on the right) to the sophisticated procedure using a sphere (on the left).*

6. Analogy as the way to 4D

Analogy is very methodically used in a fascinating French video Dimensions (Dimensions, 2008). The 2nd part of the 1st set “La dimension trois” presents an experiment of a solid passing through a “lizard plane” inspired by E.A. Abbott’s book “Flatland” [1]. This video recording first presents passing of a solid through a plane and creation of a dynamic section (Figure 7 on the left), consequently the watcher’s task is to tell the original solid passing through the plane from the dynamically changing section (Figure 7 on the right).

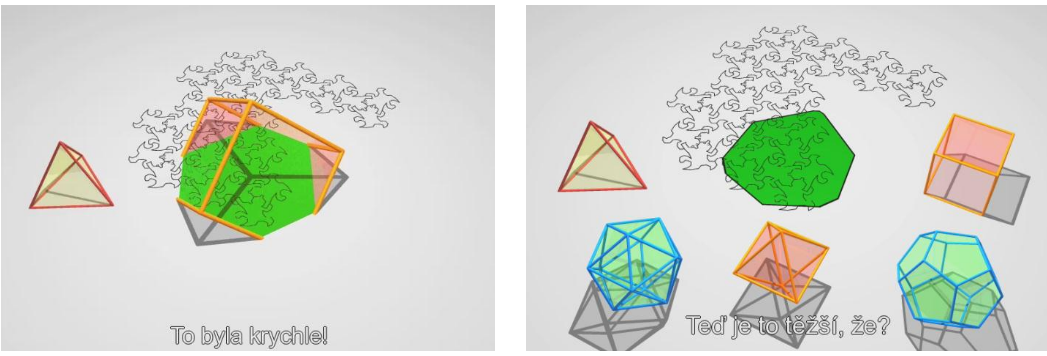

Fig. 7: *Snapshot from the video Dimensions, telling the solid passing through the plane from its section. Adapted from [5]*

This video was shown to both pre- and in-service teachers and their reactions suggest that the video offers full understanding but a more interactive and variable aid is needed if the ability "to see the higher dimension" is to be trained and developed. That is why we created several figures in Cabri 3D that simulated the same experiment for different solids passing through the lizard plane. The figure also makes it possible to watch the dynamically changing shape of the section. Moreover, it enables having a look at the situation from the side and see the position of the section in the solid. We also created a few examples with wire models, where the section is formed by a set of intersection points.

Reactions of the teachers show their surprise how difficult it is to tell the solid if it is not a regular one. We came across worst results in the set of problems in which the plane was passed through by a regular hexagonal pyramid but in various arbitrary positions (Figure 8).

This training was meant to prepare teachers for use of analogy in transition to four-dimensional space. The sequels of the video recording Dimensions carry on presentation of discovery of 4D space, which is inspiring when looking for analogy in construction problems also between 3D and 4D space. However, we do not have a corresponding modeling environment that would allow verification of some of the hypotheses. Students must rely entirely on their sense of analogy trained in previous search for analogical construction problems between 2D and 3D.

What can the wording of analogical problem and analogical solution in 4D to the above presented problems be? Is this type of task a mere linguistic practice, or may there be in the background of its solution some geometrical notion? These are the questions naturally arising when this type of tasks is used. At least they definitely evoke 4D space as space in which normal geometry takes place and comprehensible problems are assigned. Therefore pre-service teachers are introduced to the way of comprehending more dimensions of geometry.

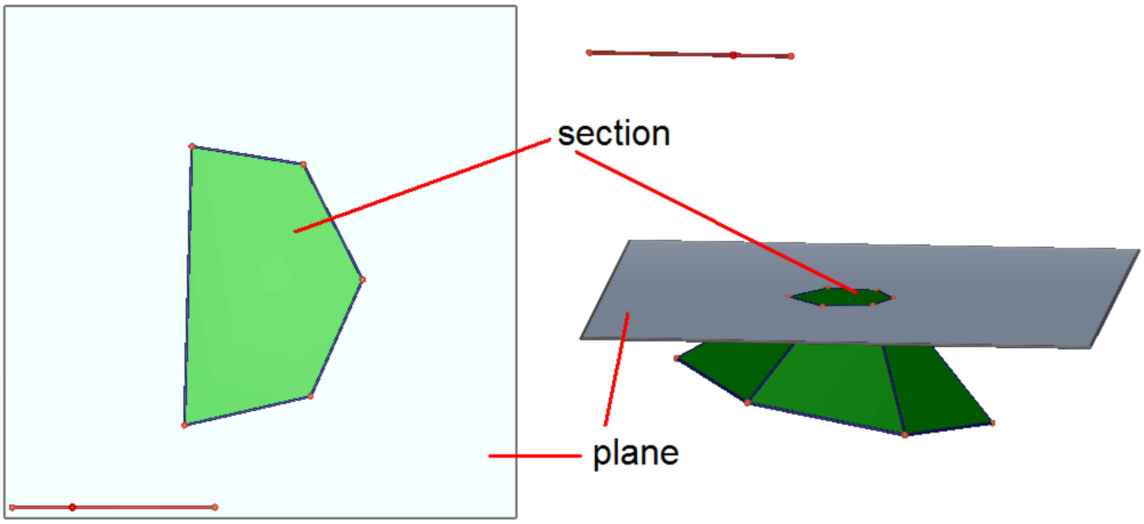

Figure 8: *Interactive figure made in Cabri 3D for the activities of telling the solid passing through a plane from its dynamic section. Both pictures show the same figure in different perspectives.*

Table 3 presents an example of such a problem in 4D in which students could proceed from 2D and in whose case they came to an agreement on its assignment and solution.

This activity was the final activity in the pre-service training course. Students found it extremely difficult to cope with this issue; they failed to find analogical assignment of a problem from 3D in 4D. If the assignment was given in 4D, in a limited extent they were able to look for analogical solutions. However, it was obvious that they lacked the needed terminology. The group of students who managed to find the construction procedure usually found it thanks to a successful grammar exercise in which they replaced the objects in higher dimension by their analogues.

3D	4D
ASSIGNMENT: Given is a tetrahedron. Construct the sphere circumscribed to this tetrahedron. **SOLUTION:** The circumcenter lies on the intersection of **planes** of symmetry of tetrahedron edges.	**ASSIGNMENT:** Given is a 4D simplex, construct the hypersphere circumscribed to this simplex. **SOLUTION:** Circumcenter lies on the intersection of **hyperplanes** of symmetry of simplex edges.

Table 3: Analogical construction problem between 3D and 4D

7. Conclusion

Pre-service teachers are able to:

- perceive analogy between planar and spatial construction
- pose problems in 3D analogical to problems in plane
- in a limited extent find analogical solutions to 2D constructions in 3D
- in a very limited extent describe or verbally generate solutions of analogical construction problems in 3D

Even when students know the analogical construction procedure in 3D, they tend to prefer more complicated construction procedures that use 2D construction steps to 3D construction steps of the analogical procedure.

Computer technologies may model situations supporting development of the ability to generalize thanks to development of the ability to find analogies between constructions in different dimensions.

When posing analogical problems in 4D we lack any modeling environment that would enable us to get insight into 4D. This can be substituted by three-dimensional environment that can help to make the students understand in some situations. This can be documented by the following example of a hypercube in 4D – for more information see [2]. Some students were able to visualize with the help of the popular model of a hypercube (Figure 9 on the left) that one 3D-cell lies inside another 3D-cell, which they found very strange because usually a solid in 3D does not have one side inside another. The lecturer’s argument that students see only the projection of the solid failed to help. It was a 3D model of the hypercube prepared by the lecturer in which optical illusion causes that the smaller cube seems to be lying inside a larger one (even though when rotated it is obviously not the case) that helped the students to understand (on the right).

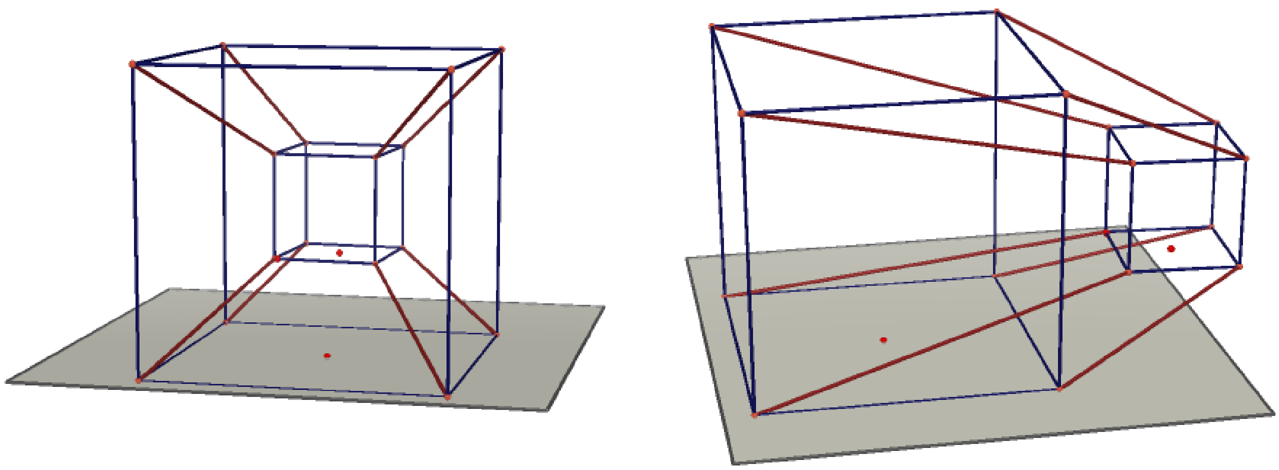

Figure 9: *Model of projection of 4D hypercube into E^3. Both pictures show the same (unchanged) figure perceived from a different perspective [10].*

8. References

[1] Abbott, E. A., *Flatland, a romance of many dimensions*, Sealy a Co., 1884. http://www.ibiblio.org/eldritch/eaa/FL.HTM

[2] Bourke, P., *Regular Polytopes (Platonic solids) in 4D.* [online] University of Western Australia, 2003. http://local.wasp.uwa.edu.au/~pbourke/geometry/platonic4d/

[3] Dell, D., Analogies Lesson by Diana Dell, Ed.S, In Dell, D. (ed.): *Teaching and Learning with Technology*, cit. 2010-06-15. <http://mrsdell.org/analogies>

[4] Hofstadter, D. R., Epilogue: Analogy as the Core of Cognition, In Gentner, D., Holyoak, K.J. & Kokinov, B. (eds.). *The Analogical Mind: Perspectives from Cognitive Science*. Cambridge, MA, MIT Press, 2001. ISBN 0-262-57139-0

[5] Leys, J., Ghys, É. & Alvarez, A., *Dimensions*, Educational video, 2008. http://www.dimensions-math.org

[6] Oldknow, A., Solid geometry and Cabri 3D, In *CY maths ICT*, Chartwell-Yorke Ltd., 2004. http://www.chartwellyorke.com/cabri3d/oldknow/solidgeometry.doc

[7] Polya, G., *Induction and Analogy in Mathematics (Mathematics and Plausible Reasoning, Vol I),* Princeton (NJ): Princeton University Press, 1954.

[8] Richland, L. E., Holyoak, K. J. & Stigler, J. W., The role of analogy in teaching middleschool mathematics, *Cognition and Instruction* (22, 37-60), 2004.

[9] Sayer, A., *Method in social science: A realist approach*, London and New York: Routledge, 1992.

[10] Vaníček, J., *Počítačové kognitivní technologie ve výuce geometrie*, Praha: Karlova univerzita, 2009. ISBN 978-80-7290-394-8.